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Chapter 8

HYPOTHESIS TESTING

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CASE STUDY:

Do the newer models of the Nissan™ Altima get more miles per gallon than previous models?

To answer the question, researchers gathered a random sample of new Nissan™ Altimas and documented the miles per gallon. After performing a hypothesis test, it was determined that there was sufficient evidence to suggest that the mpg had increased.

So, what was the methodology used to determine the results?

Well, they had to do the following:

1. Perform a hypothesis test.
2. Make inferences about the population using sample data.

In the following chapter, you will learn about these concepts and their applications.

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Section 8.1:

Steps in Hypothesis Testing

Student Learning Outcomes

By the end of the section,

1. **You will understand the steps in hypothesis testing.**
2. **You will differentiate between Type I and Type II error.**

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In Chapter 7, confidence intervals were examined. The use of confidence intervals is important in statistics because it allows researchers to construct interval estimates for population parameters. In addition to constructing confidence intervals, there are times when researchers want to prove new information by testing a claim. Hypothesis testing allows researchers to pose a hypothesis, analyze sample data and generalize the sample results to the population.

DEFINITION

Hypothesis Testing: a procedure that uses sample data to test whether a hypothesis about the value of a population parameter is true.

Note: Hypothesis testing can also be referred to as tests of significance.

Hypothesis testing is used in many applications such as medicine, biology, education and psychology to test a theory and address a plethora of questions. For example,

- The dropout rate at a local high school is above the national average. The principal and administrative staff decides to implement a new program to retain students and reduce the dropout rate. They perform a test of hypothesis to determine if the dropout rate declined.

- A manufacturing company performs weekly checks for the number of defective parts produced by the machinery. After noticing an influx of defective parts, the company implements a new process. Is there sufficient evidence to suggest that the new process is effective in reducing the number of defective parts?

- A holistic medicine practitioner believes a particular herb can increase memory. After taking the herb for 6 months, his patients were given a memory test to

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determine if their memory increased. Was the herb effective in increasing memory?

Testing a hypothesis involves five steps.

Steps for Testing a Hypothesis

Step 1: Hypotheses
Step 2: Level of Significance
Step 3: Test Statistic
Step 4: Critical Value(s) or P-value
Step 5: Decision and Conclusion

Step 1: Hypotheses – When stating the hypotheses, two statements are made.

DEFINITION

Null Hypothesis (H_0): a hypothesis about the value of the population parameter, which is assumed to be true.

Alternative Hypothesis (H_A): an alternative hypothesis about the value of the population parameter.

Note: H_0 is read as "H naught" or "H subzero" and H_A is read as "H sub-A".

The null hypothesis typically originates from information seen in a journal, newspaper, advertisement, or article, to name a few. The value observed is assumed to be true and thus, represents the population parameter. On the contrary, the alternative hypothesis is a claim that refutes the null hypothesis.

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To write the null and alternative hypotheses, researchers translate the English statements into mathematical statements.

The three possible pairs are the following:

H_0: $\mu = k$ H_A: $\mu < k$	H_0: $\mu = k$ H_A: $\mu > k$	H_0: $\mu = k$ H_A: $\mu \neq k$

Note: When analyzing proportions, replace μ with p.

k is the assumed value of the population parameter. Thus, it will be equal (=) to the population parameter when stating the null hypothesis. When stating the alternative hypothesis, <, >, or ≠ can be used. The alternative hypothesis is determined by the researcher and the goal of the study. The following chart shows key words that can be used to write mathematical statements:

Symbol	Key Words
=	Is Equal to The same as
<	Below Smaller than Less than Decreased by Fewer than
>	Above Larger than More than Increased by Greater than
≠	Different from Not equal to Not Differs from Changed from

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Once the hypotheses are formed, the next step is to determine the level of significance.

Step 2: Level of Significance – The maximum allowable probability of making a Type I error.

More than likely, some type of error will occur since the analysis is based on a sample rather than the population. Therefore, the level of significance should be as small as possible. Common levels of significance are 0.001, 0.01, 0.05 and 0.10. However, any level of significance can be selected. The level of significance is chosen by the researcher. In cases where consequences of error would be severe, it's best to select a smaller level of significance such as 0.01.

DEFINITION

Level of Significance: the probability of rejecting the null hypothesis when it is true. The level of significance is represented as α (Alpha).

After selecting a level of significance, researchers can collect and analyze the data. Once the data is collected and analyzed, the test statistic can be computed.

Step 3: Test Statistic – A standardized sample statistic.

DEFINITION

Test Statistic: a numerical summary calculated from sample data. The test statistic can vary based on the population parameter being tested as well as the sampling distribution.

The value of the test statistic may or may not be unusual or extreme. In cases where the test statistic is very unusual or

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extreme, theory suggests that the observed data is rare and provides evidence against the null hypothesis. Therefore, researchers can conclude that the null hypothesis is false.

STOP!!

Ensure the test statistic is *statistically* significant. If the results are statistically significant, then the results are unlikely to have occurred due to chance. As a result, there is statistically significant evidence to reject the null hypothesis.

The significance can be viewed by using the critical value approach or p-value approach.

Step 4: Critical Value(s) or P-value – Value(s) used in conjunction with the test statistic or level of significance to determine if the null hypothesis should or should not be rejected.

DEFINITION

Critical Value: a value created based on the level of significance and type of test (e.g. left-tailed test, right-tailed test, two-tailed test). The value is used as the boundary for the rejection region.

DEFINITION

P-value: the probability of observing the test statistic as extreme or more extreme as the one observed from the sample data, assuming the null hypothesis is true.

The critical value approach *or* p-value approach can be used for a hypothesis test. Both approaches can be used but only one approach is required.

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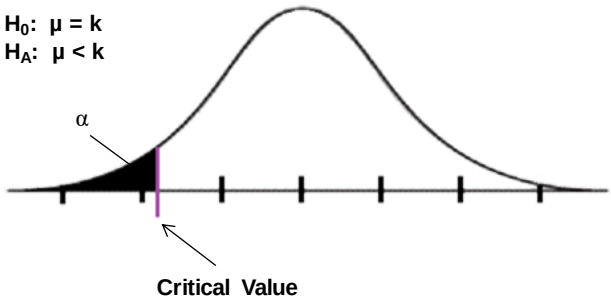
To determine the critical values, make note of the level of significance and the type of test (e.g. left-tailed test, right-tailed test, two-tailed test).

Therefore, three possibilities can be created.

1. If the alternative hypothesis contains **<**, then the study is a **left-tailed test**.



The shaded region is the rejection region. The area contained in the rejection region is equal to the level of significance.

If the test statistic is ***LESS*** than the critical value, then researchers can reject the null hypothesis. Otherwise, the null hypothesis cannot be rejected.

2. If the alternative hypothesis contains **>**, then the study is a **right-tailed test**.

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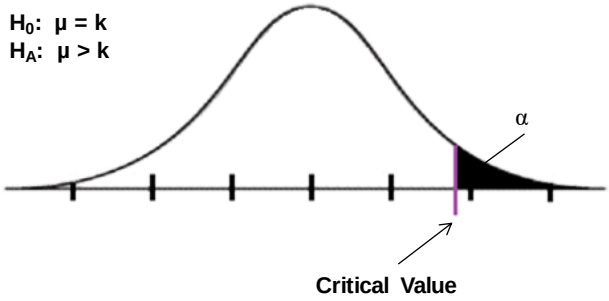
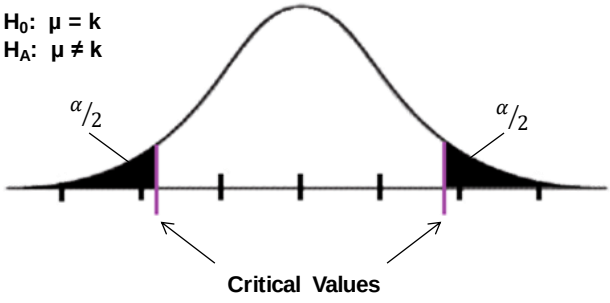


The shaded region is the rejection region. The area contained in the rejection region is equal to the level of significance.

If the test statistic is ***GREATER*** than the critical value, then researchers can reject the null hypothesis. Otherwise, the null hypothesis cannot be rejected.

Note: Left- and right-tailed tests are also considered one-tailed tests.

3. If the alternative hypothesis contains **≠**, then the study is a **two-tailed test**.



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The shaded regions are the rejection regions. The area in each rejection region is equal to half the level of significance.

If the test statistic is ***LESS*** than the lower critical value OR ***GREATER*** than the upper critical value, then researchers can reject the null hypothesis. (In a two-tailed test, the test statistic can either be positive or negative.) Otherwise, the null hypothesis cannot be rejected.

Note: Critical values will vary depending on the sampling distribution. In future chapters, X^2 (X is pronounced "ki") and F-distributions will be explored. Thus, the critical values will change.

To determine the p-value, make note of the test statistic and the type of test (e.g. left-tailed test, right-tailed test, two-tailed test).

- For a left-tailed test, the p-value is the area to the left of the test statistic.
- For a right-tailed test, the p-value is the area to the right of the test statistic.
- For a two-tailed test, the p-value is twice the area to the left of a negative test statistic **OR** twice the area to the right of a positive test statistic.

Once the p-value is computed, compare it to the level of significance. If the p-value is less than or equal to the level of significance, then the null hypothesis can be rejected. Otherwise, the null hypothesis cannot be rejected. The p-value tells researchers the statistical significance of the hypothesis test.

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Note: Small p-values provide strong evidence against the null hypothesis. Very small p-values indicate the observed test statistic is **very** unusual.

After using the critical value approach or p-value approach, a decision and conclusion can finally be stated regarding the hypothesis test.

Step 5: Decision and Conclusion – A final decision is made regarding the null hypothesis and the conclusion about the study is stated.

DEFINITION

Decision: a decision is made in regards to rejecting or not rejecting the null hypothesis.

DEFINITION

Conclusion: a summary statement of the results.

When making a decision, there are two options: reject the null hypothesis or do not reject the null hypothesis. Rejecting the null hypothesis is NOT the same as "accepting" the null hypothesis. It simply means that researchers do not have enough evidence to disprove the null hypothesis. An analogy of this concept is "guilty vs. not guilty". In a court of law, there is either enough evidence to prove guilt or there is not enough evidence to prove guilty. The verdict never states that the defendant is "innocent".

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Since sample data is being analyzed rather than population data, error is inevitable. There are two important types of error that can possibly be made.

DEFINITION

Type I error (α): error that occurs when the null hypothesis is rejected when in fact it is true.

Type II error (β): error that occurs when the null hypothesis is not rejected when in fact it is false.

Note: The probability of making a Type I error is equal to the level of significance. The probability of making a Type II error is equal to β (Beta). β is not equal to $1 - \alpha$. The computation of β is beyond the scope of this course.

Although error is inevitable, there are times when the correct decision is made. The following table shows the four possible conclusions of a hypothesis test.

Decision	**H_0 is True**	**H_0 is False**
Reject H_0	Type I Error	Correct Decision
Do Not Reject H_0	Correct Decision	Type II Error

Using the decision, researchers can state the conclusion of the hypothesis test. When stating the conclusion, it's best to speak in terms of the alternative hypothesis. Recall, the purpose of hypothesis testing is to test a claim, which is typically the alternative hypothesis. The goal is to prove new information. Therefore, researchers either have or do not have enough evidence to support the alternative hypothesis. The conclusion can be written as one of the two following statements:

- "There is sufficient evidence to suggest H_A." (Use this statement when the null hypothesis is rejected.)

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- "There is *not* sufficient evidence to suggest H_A." (Use this statement when the null hypothesis is *not* rejected.)

Note: Replace H_A with the English statement that represents the alternative hypothesis.

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Section 8.2:

Z-Test

Student Learning Outcomes

By the end of the section,

1. **You will conduct a hypothesis test for a population mean with σ known.**

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Recall, the purpose of hypothesis testing is to hopefully prove new information. Researchers can use sample data to reject or not reject the null hypothesis. As a result, they will be able to make an inference about the population parameter of interest.

By knowing the five steps in hypothesis testing, researchers can conduct a test of hypothesis for a population mean given σ is known. Using the same underlying condition that was used to construct a z-interval, researchers can conduct a test of hypothesis. It will be referred to as a **z-test**. A z-test follows the same sampling distribution as a z-interval.

Therefore, the following procedure can be used to conduct a z-test for μ given σ is known:

Procedure for Conducting a Z-Test

1. **Hypotheses:** ***Select the appropriate hypotheses from the following 3 options…***

H_0: $\mu = k$	H_0: $\mu = k$	H_0: $\mu = k$
H_A: $\mu < k$	H_A: $\mu > k$	H_A: $\mu \neq k$

2. **Level of Significance**

3. **Test Statistic:** $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

4. **Critical Value(s) or P-value**

5. **Decision and Conclusion**

For practice, let's explore three examples!

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Example 1. In a recent study, it was shown that the average application fee for colleges is $25. A high school student who's been applying to colleges believes this amount is not correct and the true average is different from $25. So, he decides to perform a test of hypothesis at the 5% level of significance. He randomly samples 36 colleges. From the sample, the mean is computed to be $28. Assume the population standard deviation is $18. At $\alpha = 0.05$, is there sufficient evidence to suggest that the student is correct?

Step 1: Hypotheses. Recall, the null hypothesis is derived from a previous study, article, journal, etc. It is a statement assumed to be true. For this example, the null hypothesis is the following: the average application fee is equal to $25. Using notation, H_0: $\mu = 25$. When writing the null hypothesis, ensure to select the appropriate population parameter. μ is used because the "*average*" application fee is being analyzed.

The alternative hypothesis is a statement about the researcher's claim. For this example, the high school student would like to prove the average application fee is different from $25. Using notation, H_A: $\mu \neq 25$. Please note that $\neq$ is used because the high school student believes the average application fee is "different" from $25. Thus, the hypotheses are the following:

H_0: $\mu = 25$
H_A: $\mu \neq 25$

Step 2: Level of Significance. The level of significance is 5%. Thus, $\alpha = 0.05$.

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Step 3: Test Statistic. Using the values from the scenario, make note of the values that represent the sample mean and population mean. Be careful not to confuse the two values. The sample mean is derived from sample data. Thus,

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$z = \frac{28 - 25}{18 / \sqrt{36}}$$

$$z = 1$$

The calculation shows the test statistic is equal to 1. Therefore, the sample results deviate from the population mean by one standard error. An illustration of the comparison between the sample mean and population mean can be seen on the following page. The first illustration shows the original distribution (i.e. the sampling distribution of the sample mean with $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$). The second illustration shows its conversion to a standard normal distribution.

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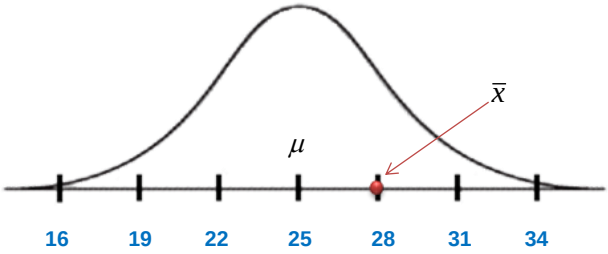

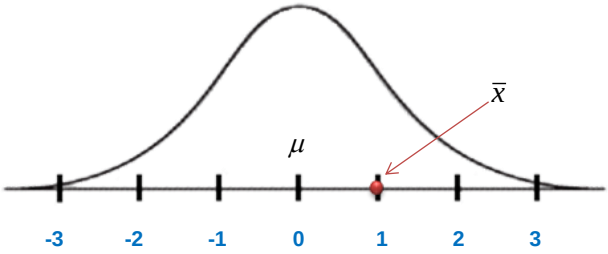

Note: Recall, the test statistic is a numerical summary based on sample data. The test statistic is used to determine if there is or is not enough evidence to reject the null hypothesis, the assumed value of the population parameter. The sample mean will be close to the population mean if the null hypothesis is correct. However, if it is extremely unusual or different from the population mean, it provides evidence against the null hypothesis. Thus, allowing researchers to reject the null hypothesis.

Based on the illustrations, there is a difference between the population mean and the sample mean. However, is the difference **statistically** significant? To answer the question, researchers use the critical value approach or p-value approach to make a decision.

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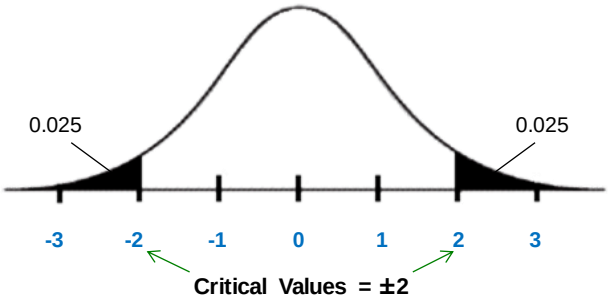
Step 4: Critical Values or P-value. When performing a hypothesis test, use either the critical value approach **OR** p-value approach.

Critical Value Approach: To use the critical value approach, make note of the level of significance and the type of test (e.g. left-tailed test, right-tailed test, two-tailed test). For this example, $\alpha = 0.05$ and it's a two-tailed test. Therefore, this is a two-tailed test with area of 0.025 in each tail.

Since the sampling distribution follows a normal distribution, the Empirical Rule or Standard Normal Table can be used to find the critical values. If the Empirical Rule is used, the z critical values can be approximated to be ±2 because there is 2.5% in each tail. Therefore, the rejection regions can be illustrated as the following:



If the Standard Normal Table is used, the precise z critical values can be determined. The precise z critical values can be seen in the following table for the common significance levels. The z critical values for left-tailed, right-tailed and two-tailed tests. *For other levels of significance, refer to the Standard Normal Table.*

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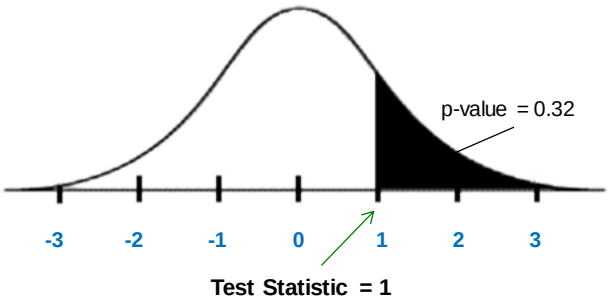
α	Type of Hypothesis Test		
	Left-tailed <	Right-tailed >	Two-tailed ≠
0.10	−1.28	1.28	±1.645
0.05	−1.645	1.645	±1.96
0.02	−2.05	2.05	±2.33
0.01	−2.33	2.33	±2.575
0.001	−3.09	3.09	±3.29

The table shows the precise z critical values for $\alpha = 0.05$ to be ±1.96, which is close to ±2.

P-value Approach: For a two-tailed test, the p-value is either twice the area to the left of a negative test statistic or twice the area to the right of a positive test statistic. This example shows the test statistic to be positive. Therefore, using the Empirical Rule, the area to the right of 1 is 0.16. After multiplying it by two, the p-value is equal to 0.32.



If the Standard Normal Table is used, the p-value is 0.317, which is close to 0.32.

Note: Either the critical value approach or the p-value approach can be used. Both approaches are not required. The two

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approaches are shown for illustrative purposes only. The researcher determines which approach is used.

For a review about critical values and p-values, view Section 8.1: Steps in Hypothesis Testing.

Step 5: Decision and Conclusion. To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: Now that the z critical values are known, compare them to the value of the test statistic to make a decision regarding whether to reject or not reject the null hypothesis. For this example, the test statistic is 1. Since the value is positive, compare it to the upper critical value which is equal to 1.96. Recall, to make a decision to reject the null hypothesis, a positive test statistic must be greater than the upper critical value or a negative test statistic must be less than the lower critical value for a two-tailed test. After making the comparison, the value of the test statistic is NOT greater than the upper critical value. Thus, the null hypothesis cannot be rejected.

P-value Approach: The p-value was determined to be 0.32. This value means that about 32% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. 32% is not small enough to be considered an unusual event. To make a decision to reject the null hypothesis, the p-value must be less than or equal to the level of significance. The level of significance for this example is 0.05. The p-value is NOT less than or equal to the level of significance. Thus, the null hypothesis cannot be rejected.

Note: Regardless of whether the critical value or p-value approach is used, the decision will be the same.

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Both approaches resulted in a decision to not reject the null hypothesis. Now that the decision has been made, the conclusion can be stated.

"There is not sufficient evidence to suggest that the average application fee for colleges is different from $25."

For a review about making a decision and stating a conclusion, view Section 8.1: Steps in Hypothesis Testing.

Example 2. In 2002, it was reported that the average cost of orthodontic treatment for children was $2000. *Has orthodontic treatment for children increased significantly since 2002?* An assistant in an orthodontics' office performs a test of hypothesis at $\alpha = 0.01$ to answer this question. He samples 100 patients under the age of 18 and finds the mean to be $2150. Assume the population standard deviation is $500. At $\alpha = 0.01$, is there sufficient evidence to suggest that the average cost of orthodontic treatment for children has increased since 2002?

Step 1: Hypotheses. The null hypothesis is derived from a previous study, article, journal, etc. It is a statement assumed to be true. For this example, the null hypothesis is the following: the average cost of orthodontic treatment for children is equal to $2000. Using notation, H_0: $\mu = 2000$.

The alternative hypothesis is a statement about the researcher's claim. For this example, the assistant would like to determine if the average cost of orthodontic treatment for children is more than $2000. Using notation, H_A: $\mu > 2000$. Please note that > is used because the assistant would like to show the average

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cost of orthodontic treatment for children has "increased". Thus, the hypotheses are the following:

H_0: $\mu = 2000$
H_A: $\mu > 2000$

Step 2: **Level of Significance.** Because $\alpha = 0.01$, the level of significance is 1%.

Step 3: **Test Statistic.** Using the values from the scenario, make note of the values that represent the sample mean and population mean. Be careful not to confuse the two values. The sample mean is derived from sample data. Thus,

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$z = \frac{2150 - 2000}{500 / \sqrt{100}}$$

$$z = 3$$

Step 4: **Critical Value or P-value.** When performing a hypothesis test, use either the critical value approach **OR** p-value approach.

Critical Value Approach: For this example, $\alpha = 0.01$ and it's a right-tailed test. Therefore, this is a one-tailed test with an area of 0.01 in the right tail. Since the precise z critical values from the Standard Normal Table are preferred over the approximated

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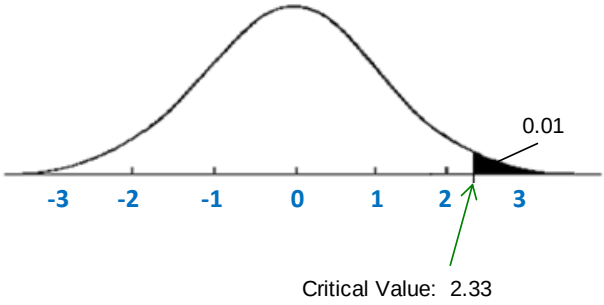
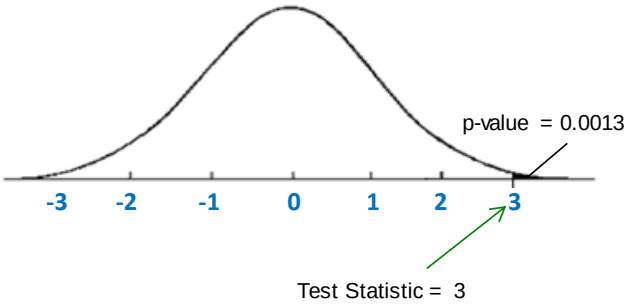
values of the Empirical Rule, the z critical value is 2.33. The rejection region is illustrated below.



P-value Approach: For a right-tailed test, the p-value is the area to the right of the test statistic. Using the Standard Normal Table, the p-value is 0.0013.



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Step 5: Decision and Conclusion. To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. For a right-tailed test, if the test statistic is greater than the critical value, the null hypothesis can be rejected. For this example, the test statistic of 3 is greater than the z critical value of 2.33. Thus, the null hypothesis can be rejected.

P-value Approach: The p-value was determined to be 0.0013. This value means that about 0.13% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis can be rejected because the p-value is $\leq \alpha$.

Since the null hypothesis can be rejected, the conclusion is the following:

"There is sufficient evidence to suggest that the average cost of orthodontic treatment for children is greater than $2000."

Example 3. In a local newspaper, it was reported that the national average to rent a 1-bedroom apartment is $660 per month. After reading this, a county official believes the cost is less for residents in her county. To test this claim, she samples 64 complexes which have 1-bedroom apartments. From the sample, the mean is determined to be $640. Assume the population standard deviation is $160. Is there sufficient evidence to support the county official's claim at the 10% level of significance?

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Step 1: Hypotheses. The newspaper reports the national average to rent a 1-bedroom apartment is $660. Using notation, H_0: μ = 660.

The alternative hypothesis is a statement about the researcher's claim. For this example, the county official would like to determine if the average cost to rent a 1-bedroom apartment in her county is less than $660. Using notation, H_A: μ < 660. Please note that < is used because the county official believes the average cost to rent a 1-bedroom apartment in her county is "less than" the national average. Thus, the hypotheses are the following:

H_0: μ = 660
H_A: μ < 660

Step 2: Level of Significance. The level of significance is 10%. Thus, α = 0.10.

Step 3: Test Statistic. Using the values from the scenario, make note of the values that represent the sample mean and population mean. Be careful not to confuse the two values. The sample mean is derived from sample data. Thus,

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$z = \frac{640 - 660}{160 / \sqrt{64}}$$

$$z = -1$$

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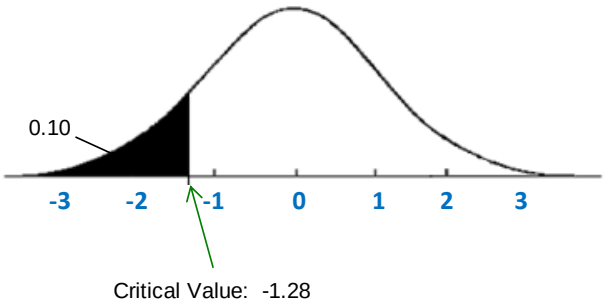
Step 4: Critical Value or P-value. When performing a hypothesis test, use either the critical value approach **OR** p-value approach.

Critical Value Approach: For this example, $\alpha = 0.10$ and it's a left-tailed test. Therefore, this is a one-tailed test with an area of 0.10 in the left tail. Using the Standard Normal Table, the precise z critical value is -1.28. The rejection region is illustrated below.



P-value Approach: For a left-tailed test, the p-value is the area to the left of the test statistic. Using the Standard Normal Table, the p-value is 0.1587.

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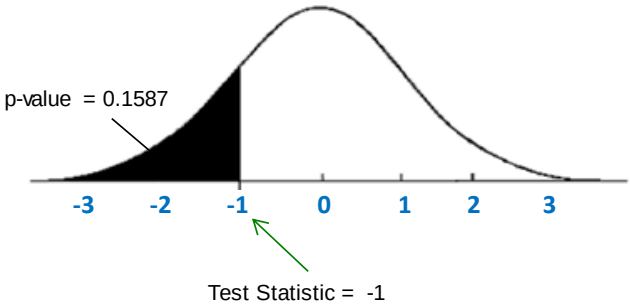

Step 5: **Decision and Conclusion.** To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. For a left-tailed test, if the test statistic is less than the critical value, the null hypothesis can be rejected. For this example, the test statistic of -1 is NOT less than the z critical value of -1.28. Thus, the null hypothesis cannot be rejected.

P-value Approach: The p-value was determined to be 0.1587. This value means that about 15.87% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, unfortunately, the null hypothesis cannot be rejected because the p-value is not $\leq \alpha$.

Since the null hypothesis cannot be rejected, the conclusion is the following:

"There is not sufficient evidence to suggest that the average rent of a 1-bedroom apartment in the county is less than $660."

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In this section, tests of hypothesis were performed to test a population mean with σ, the population standard deviation, known. However, in many real-world applications, the population standard deviation is rarely known. If the population standard deviation is not known, a z-test should not be performed. Researchers can test a population mean with σ unknown using a **t-test**, which will be discussed in the next section.

Whether a z-test or t-test is performed, underlying conditions must be met. For all tests of hypothesis including those covered in upcoming sections and chapters, the assumptions are the following: the population is normal and samples are randomly selected. If the distribution is not normal, **nonparametric tests** must be used. Nonparametric tests do not require the population's distribution to meet underlying conditions such as normality. Nonparametric statistics is beyond the scope of this course.

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Section 8.3:

T-Test

Student Learning Outcomes

By the end of the section,

1. **You will conduct a hypothesis test for a population mean with σ unknown.**

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There are times when researchers want to test a population mean but σ is unknown. For situations such as these, a **t-test** should be used. The same underlying conditions that were applied to the t-interval apply here as well.

To perform a t-test, follow the five-step procedure for conducting a test of hypothesis. The test statistic for a t-test will follow a t-distribution with degrees of freedom = n – 1. Thus, the t-distribution will be used to determine the critical values for the rejection regions. The following shows the procedure for conducting a t-test:

Procedure for Conducting a T-Test

1. **Hypotheses:** ***Select the appropriate hypotheses from the following 3 options…***

H_0: $\mu = k$	H_0: $\mu = k$	H_0: $\mu = k$
H_A: $\mu < k$	H_A: $\mu > k$	H_A: $\mu \neq k$

2. **Level of Significance**

3. **Test Statistic:** $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ **with df = n – 1**

4. **Critical Value(s) or P-value**

5. **Decision and Conclusion**

Let's explore examples!

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Example 1. A travel magazine reports the average price of a hotel room is $87 per night. A frequent traveler believes the average price is much higher in Miami. She samples 7 hotels and determines the mean to be $135 with a standard deviation of $40. Assume the sample is taken from a normal population. Is there sufficient evidence to suggest that the average price for a hotel room in Miami is greater than $87 per night at $\alpha = 0.05$?

Step 1: Hypotheses. The travel magazine reports the average price of a hotel room is $87 per night. Using notation, H_0: $\mu = 87$.

A traveler believes the average price is much higher in Miami. Using notation, H_A: $\mu > 87$. Thus, the hypotheses are the following:

H_0: $\mu = 87$
H_A: $\mu > 87$

Step 2: Level of Significance. $\alpha = 0.05$. Thus, the level of significance is 5%.

Step 3: Test Statistic. The test statistic is the following:

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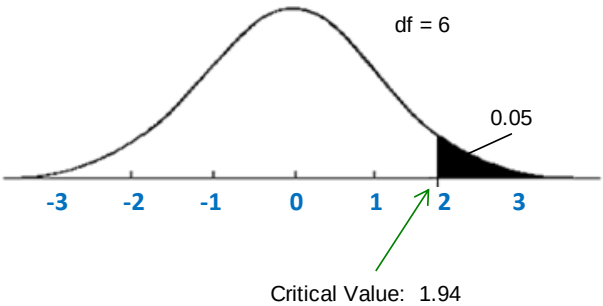
$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$t = \frac{135 - 87}{40/\sqrt{7}}$$

$$t = 3.17$$

Step 4: Critical Value or P-value.

Critical Value Approach: For this example, α = 0.05 and it's a right-tailed test. Therefore, this is a one-tailed test with an area of 0.05 in the right tail. Using the **t-distribution**, the t critical value is 1.94 based on 6 degrees of freedom. Thus, the rejection region is the following:



P-value Approach: For a right-tailed test, the p-value is the area to the right of the test statistic. It's rather difficult to use the t-distribution table to provide an **exact** p-value. Therefore,

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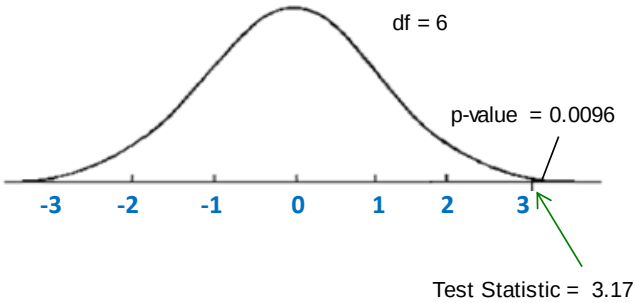
technology must be used to compute p-values for t-distributions. To see an example of how to use technology to compute the p-value, view the following video: https://youtu.be/l_nuKYiBrYk

Using technology, the p-value is 0.0096.



Step 5: Decision and Conclusion. To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. For a right-tailed test, if the test statistic is greater than the critical value, the null hypothesis can be rejected. For this example, the test statistic of 3.17 is greater than the t critical value of 1.94. Thus, the null hypothesis can be rejected.

P-value Approach: The p-value was determined to be 0.0096. This value means that about 0.96% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis can be rejected because the p-value is $\leq \alpha$.

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Since the null hypothesis can be rejected, the conclusion is the following:

"There is sufficient evidence to suggest that the average price of a hotel room in Miami is greater than $87 per night."

Example 2. In a journal published early this year, the average amount of water used per day per person was 123 gallons. A water conservationist believes he uses less water per day and would like to test his theory. He samples 25 days and determines the average amount of water used to be 111 gallons with a standard deviation of 24 gallons. Assume the sample is taken from a normal population. At the 1% level of significance, is there sufficient evidence to support the water conservationist theory?

Step 1: Hypotheses. The journal reported the average amount of water used per person per day was 123 gallons. Using notation, H_0: $\mu = 123$.

The water conservationist believes he uses less water, on average, per day. Using notation, H_A: $\mu < 123$. Thus, the hypotheses are the following:

H_0: $\mu = 123$
H_A: $\mu < 123$

Step 2: Level of Significance. The level of significance is 1%. Thus, $\alpha = 0.01$.

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Step 3: Test Statistic. The test statistic is the following:

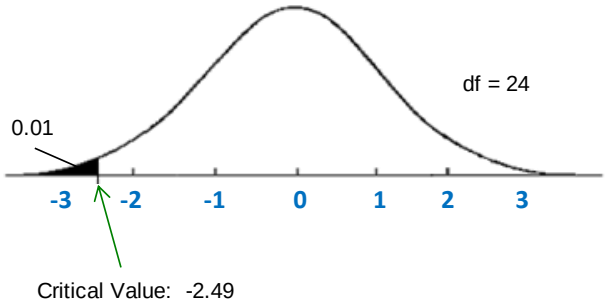
$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$t = \frac{111 - 123}{24/\sqrt{25}}$$

$$t = -2.5$$

Step 4: Critical Value or P-value.

Critical Value Approach: For this example, α = 0.01 and it's a left-tailed test. *Note*: When using the **t-distribution table**, the t critical values shown for one-tailed tests are for tests with area in the right tail. Thus, the values are positive. However, since the t-distribution is symmetrical, it can be implied that the t critical values for one-tailed tests with area in the left tail are the negative t critical values. Thus, the t critical value is −2.49 based on 24 degrees of freedom. The rejection region is the following:



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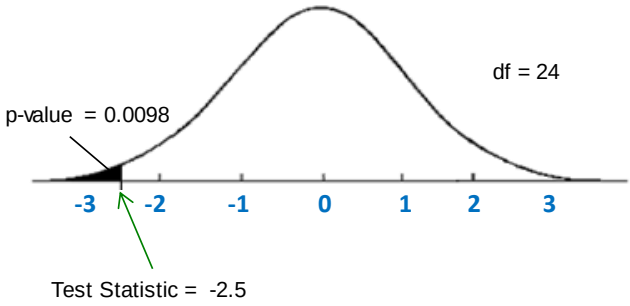
P-value Approach: Using technology, the p-value is 0.0098.



Step 5: **Decision and Conclusion.** To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. For a left-tailed test, if the test statistic is less than the critical value, the null hypothesis can be rejected. For this example, the test statistic of −2.5 is less than the t critical value of −2.49. Thus, the null hypothesis can be rejected.

P-value Approach: The p-value was determined to be 0.0098. This value means that about 0.98% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis can be rejected because the p-value is $\leq \alpha$.

Since the null hypothesis can be rejected, the conclusion is the following:

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"There is sufficient evidence to suggest that the average amount of water used per day by the water conservationist is less than 123 gallons."

Example 3. A study shows that the average cost of a hospital stay for five days in Hawaii is $22,600. A medical doctor who practices in the state of Georgia believes the average cost is different in his state. He samples 9 patients whose length of stay in a hospital was five days and documents the cost. The mean is computed to be $17,400 with a standard deviation of $5250. Assume the sample is taken from a normal population. At $\alpha = 0.10$, is there sufficient evidence to support the medical doctor's claim?

Step 1: Hypotheses. The study shows the average cost of a hospital stay for five days in Hawaii is $22,600. Using notation, H_0: $\mu = 22600$.

A medical doctor believes the average cost is different in his state. Using notation, H_A: $\mu \neq 22600$. Thus, the hypotheses are the following:

H_0: $\mu = 22600$
H_A: $\mu \neq 22600$

Step 2: Level of Significance. $\alpha = 0.10$. Thus, the level of significance is 10%.

Step 3: Test Statistic. The test statistic is the following:

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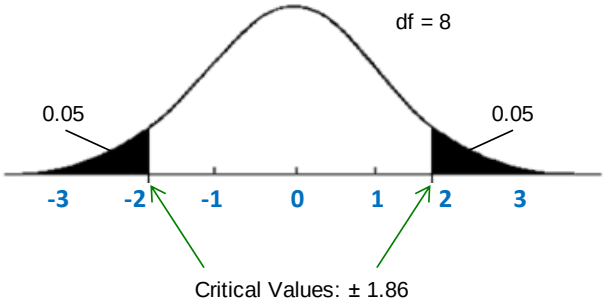
$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$t = \frac{17400 - 22600}{5250/\sqrt{9}}$$

$$t = -2.97$$

Step 4: Critical Values or P-value.

Critical Value Approach: For this example, α = 0.10 and it's a two-tailed test. Therefore, this is a two-tailed test with area of 0.05 in each tail. Recall, for a two-tailed test, alpha is divided by 2. The t critical values are ±1.86 based on 8 degrees of freedom. Thus, the rejection regions are the following:



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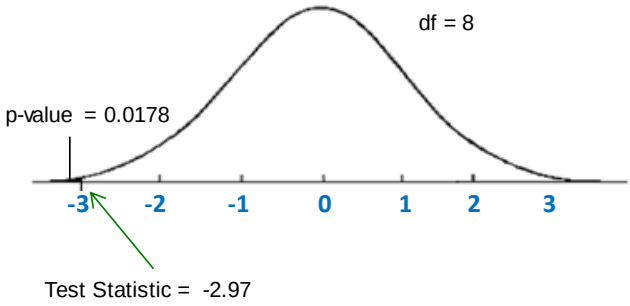
P-value Approach: Using technology, the p-value is 0.0178.



Step 5: **Decision and Conclusion.** To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. Recall, to make a decision to reject the null hypothesis, a positive test statistic must be greater than the upper critical value or a negative test statistic must be less than the lower critical value for a two-tailed test. For this example, the test statistic of −2.97 is less than the lower t critical value of −1.86. Thus, the null hypothesis can be rejected.

P-value Approach: The p-value was determined to be 0.0178. This value means that about 1.78% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis can be rejected because the p-value is $\leq \alpha$.

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Since the null hypothesis can be rejected, the conclusion is the following:

"There is sufficient evidence to suggest that the average cost of a hospital stay for five days in Georgia is different from $22,600."

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Section 8.4:

Proportion Z-Test

Student Learning Outcomes

By the end of the section,

1. **You will conduct a hypothesis test for a population proportion.**

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There are times when researchers would like to make inferences about a population proportion rather than a population mean. In this section, hypothesis testing for a population proportion will be explored. This type of test will be referred to as a **proportion z-test**.

The proportion z-test follows the same sampling distribution as a proportion z-interval. For a review, refer to Section 7.3: Proportion Z-Interval. Using the same underlying conditions that were used to construct a proportion z-interval, researchers can conduct a test of hypothesis. The procedure for conducting a proportion z-test is the following:

Procedure for Conducting a Proportion Z-Test

1. **Hypotheses:** ***Select the appropriate hypotheses from the following 3 options…***

H_0: $p = k$	H_0: $p = k$	H_0: $p = k$
H_A: $p < k$	H_A: $p > k$	H_A: $p \neq k$

2. **Level of Significance**

3. **Test Statistic:** $z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}}$

4. **Critical Value(s) or P-value**

5. **Decision and Conclusion**

Let's explore examples!

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Example 1. In 1999, it was reported that 30% of students at a local high school smoked cigarettes. To encourage students to stop smoking, a high school principal implemented a campaign in hopes to decrease the percentage of students who smoke. Four years later, he sampled 300 students and determined that 76 of them smoked. At the 5% level of significance, was there sufficient evidence to show the "stop smoking" campaign reduced the proportion of students at the local high school who smoked cigarettes?

Step 1: Hypotheses. In 1999, it was reported that 30% of students at a local high school smoked cigarettes. Using notation, H_0: $p = 0.30$. *Note*: ALWAYS convert the percent to a decimal.

The principal wanted to show if the campaign was effective at reducing the proportion of students at the local high school who smoked cigarettes. Using notation, H_A: $p < 0.30$. Thus, the hypotheses are the following:

H_0: $p = 0.30$
H_A: $p < 0.30$

Step 2: Level of Significance. The level of significance is 5%. Thus, $\alpha = 0.05$.

Step 3: Test Statistic. Using the values from the scenario, make note of the values that represent the sample proportion and population proportion. Be careful not to confuse the two values. The sample proportion is derived from sample data. In a sample of 300 students, 76 of them smoked cigarettes.

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Recall, $\hat{p} = \frac{x}{n}$

Thus, the sample proportion is the following:

$$\hat{p} = \frac{x}{n} = \frac{76}{300} = 0.2533$$

Moreover, the test statistic is the following:

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

$$z = \frac{0.2533 - 0.30}{\sqrt{\frac{(0.30)(0.70)}{300}}}$$

$$z = -1.76$$

Step 4: Critical Value or P-value.

Critical Value Approach: For this example, $\alpha = 0.05$ and it's a left-tailed test. Therefore, this is a one-tailed test with an area of 0.05 in the left tail. Recall, from *Section 8.2: Z-Test*, the common z critical values are the following:

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	Type of Hypothesis Test		
	Left-tailed	Right-tailed	Two-tailed
α	<	>	≠
0.10	−1.28	1.28	±1.645
0.05	−1.645	1.645	±1.96
0.02	−2.05	2.05	±2.33
0.01	−2.33	2.33	±2.575
0.001	−3.09	3.09	±3.29

Thus, the z critical value is −1.645.

P-value approach: For a left-tailed test, the p-value is the area to the left of the test statistic. Thus, the p-value is 0.039.

Step 5: Decision and Conclusion. To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. For a left-tailed test, if the test statistic is less than the critical value, the null hypothesis can be rejected. For this example, the test statistic of −1.76 is less than the z critical value of −1.645. Thus, the null hypothesis can be rejected.

P-value Approach: The p-value was determined to be 0.039. This value means that about 3.9% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis can be rejected because the p-value is ≤ α.

Since the null hypothesis can be rejected, the conclusion is the following:

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"There is sufficient evidence to suggest that the proportion of local high school students who smoke cigarettes is less than 30%."

Example 2. Research has found that 65% of high school graduates go to college. In a poverty-stricken area, a sociologist believes the percentage is different at the local high school. She samples 200 recent high school graduates and finds that 60% of them are going to college. Using a test of hypothesis, is she correct at α = 0.01?

Step 1: Hypotheses. Research has found that 65% of high school graduates go to college. Using notation, H_0: p = 0.65.

A sociologist believes the percentage is different at the local high school. Using notation, H_A: p ≠ 0.65. Thus, the hypotheses are the following:

H_0: p = 0.65
H_A: p ≠ 0.65

Step 2: Level of Significance. α = 0.01. Thus, the level of significance is 1%.

Step 3: Test Statistic. Using the values from the scenario, make note of the values that represent the sample proportion and population proportion. Be careful not to confuse the two values. The sample proportion is derived from sample data. In a sample of 200 recent high school graduates, 60% of them are going to college. Thus, the sample proportion is the following:

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$$\hat{p} = 0.60$$

Moreover, the test statistic is the following:

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

$$z = \frac{0.60 - 0.65}{\sqrt{\frac{(0.65)(0.35)}{200}}}$$

$$z = -1.48$$

Step 4: Critical Values or P-value.

Critical Value Approach: For this example, $\alpha = 0.01$ and it's a two-tailed test. Therefore, this is a two-tailed test with area of 0.005 in each tail. Thus, the z critical values are ±2.575.

P-value approach: For a two-tailed test, the p-value is either twice the area to the left of a negative test statistic or twice the area to the right of a positive test statistic. This example shows the test statistic to be negative. Therefore, the area to the left of −1.48 is 0.069. After multiplying it by two, the p-value is equal to 0.138.

Step 5: Decision and Conclusion. To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. Recall, to make a

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decision to reject the null hypothesis, a positive test statistic must be greater than the upper critical value or a negative test statistic must be less than the lower critical value for a two-tailed test. For this example, the test statistic of -1.48 is not less than the lower z critical value of -2.575. Thus, the null hypothesis cannot be rejected.

P-value Approach: The p-value was determined to be 0.138. This value means that about 13.8% of all samples would produce a test statistic at least as extreme as the one calculated if the null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis cannot be rejected because the p-value is not $\leq \alpha$.

Since the null hypothesis cannot be rejected, the conclusion is the following:

"There is not sufficient evidence to suggest that the proportion of high school graduates from the local high school in the poverty-stricken area who go to college is different from 65%."

Example 3. The use of cell phones has exponentially grown in the past decade. Nowadays, even children own a cell phone. In a recent report, the percent of elementary-aged children who own a cell phone is 25%. A mother hears this and believes the percentage of children who own a cell phone at her son's elementary school is much higher than this. She samples 350 students at the elementary school and finds that 100 of them own a cell phone. At $\alpha = 0.10$, is the mother correct?

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Step 1: Hypotheses. A report shows the percentage of elementary-aged children who own a cell phone is 25%. Using notation, H_0: p = 0.25.

A mother believes the percentage of children who own a cell phone at her son's elementary school is much higher than 25%. Using notation, H_A: p > 0.25. Thus, the hypotheses are the following:

H_0: p = 0.25
H_A: p > 0.25

Step 2: Level of Significance. $\alpha = 0.10$. Thus, the level of significance is 10%.

Step 3: Test Statistic. Using the values from the scenario, make note of the values that represent the sample proportion and population proportion. Be careful not to confuse the two values. The sample proportion is derived from sample data. In a sample of 350 students at the elementary school, 100 of them own a cell phone.

Recall, $\hat{p} = \frac{x}{n}$

Thus, the sample proportion is the following:

$$\hat{p} = \frac{x}{n} = \frac{100}{350} = 0.2857$$

Moreover, the test statistic is the following:

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$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

$$z = \frac{0.2857 - 0.25}{\sqrt{\frac{(0.25)(0.75)}{350}}}$$

$$z = 1.54$$

Step 4: Critical Value or P-value.

Critical Value Approach: For this example, $\alpha = 0.10$ and it's a right-tailed test. Therefore, this is a one-tailed test with an area of 0.10 in the right tail. Thus, the z critical value is 1.28.

P-value approach: For a right-tailed test, the p-value is the area to the right of the test statistic. Thus, the p-value is 0.061.

Step 5: Decision and Conclusion. To make a decision, either compare the critical value to the test statistic **OR** compare the p-value to the level of significance.

Critical Value Approach: For this approach, compare the value of the test statistic to the critical value. For a right-tailed test, if the test statistic is greater than the critical value, the null hypothesis can be rejected. For this example, the test statistic of 1.54 is greater than the z critical value of 1.28. Thus, the null hypothesis can be rejected.

P-value Approach: The p-value was determined to be 0.061. This value means that about 6.1% of all samples would produce a test statistic at least as extreme as the one calculated if the

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null hypothesis is true. By comparing the p-value to the level of significance, the null hypothesis can be rejected because the p-value is $\leq \alpha$.

Since the null hypothesis can be rejected, the conclusion is the following:

"There is sufficient evidence to suggest that the proportion of children at the son's elementary school who own a cell phone is greater than 25%."

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